Lesson 13. Determinants

1 Overview

- Last lesson: solving systems of linear equations, finding inverses with elementary row operations
- This lesson: detecting nonsingular/singular matrices, another way of solving systems of linear equations

2 The determinant

- The **determinant** $|A|$ of square matrix A is a uniquely defined scalar associated with A
- If $A = [a]$, then $|A| =$
- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $|A| =$

Example 1. Let $A = \begin{bmatrix} 10 & 4 \\ 8 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & -1 \end{bmatrix}$. What is $|A|$? What is $|B|$?

- What about larger matrices (3×3, 4×4, 100×100...)?
- We can use **Laplace expansion**

3 Computing the determinant for larger matrices – Laplace expansion

- Let's consider an $n \times n$ matrix A:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

- The **minor** $|M_{ij}|$ of a_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and the jth column

Example 2. Let $A = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 0 & 3 \\ 7 & 0 & -3 \end{bmatrix}$.

$|M_{11}|$ is the determinant of $\begin{bmatrix} 5 & 1 & 6 \\ 2 & 0 & 3 \\ 7 & 0 & -3 \end{bmatrix} =$, which is

$|M_{22}|$ is the determinant of $\begin{bmatrix} 5 & 1 & 6 \\ 2 & 0 & 3 \\ 7 & 0 & -3 \end{bmatrix} =$, which is

- To find the determinant of an $n \times n$ matrix, we can **expand** along any column or row:
  - Expansion along the ith row: $|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |M_{ij}|$
  - Expansion along the jth column: $|A| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} |M_{ij}|$
  - $(-1)^{i+j}$ puts a positive or negative sign in front of $a_{ij}|M_{ij}|$
    - If $i + j$ is even, then $(-1)^{i+j} =$
    - If $i + j$ is odd, then $(-1)^{i+j} =$

Example 3. Find $|A|$ using the matrix A given in Example 2 by expanding along the first row.

Example 4. Find $|A|$ using the matrix A given in Example 2 by expanding along the second column.

- A good strategy: expand along a row or column with a lot of zeros!
- The signs on the $a_{ij}|M_{ij}|$ terms form a checkerboard pattern:

4 Determinants and inverses

- A square matrix A is nonsingular if and only if $|A| \neq 0$

Example 5. Is A given in Example 2 invertible?

5 Basic properties of determinants

- Let A be a square matrix

Property I. $|A^T| = |A|$

Property II. If we interchange any two rows (or any two columns) of A, then the determinant of the new matrix will be $-|A|$

Property III. If we multiply any one row (or any one column) of A by a scalar k, then the determinant of the new matrix will be $k|A|$

Example 6. Recall that $\begin{vmatrix} 5 & 1 & 6 \\ 2 & 0 & 3 \\ 7 & 0 & -3 \end{vmatrix} = 27$. Compute the following.

$$\begin{vmatrix} 5 & 2 & 7 \\ 1 & 0 & 0 \\ 6 & 3 & -3 \end{vmatrix} = \qquad \begin{vmatrix} 5 & 6 & 1 \\ 2 & 3 & 0 \\ 7 & -3 & 0 \end{vmatrix} = \qquad \begin{vmatrix} 15 & 3 & 18 \\ 4 & 0 & 6 \\ 7 & 0 & -3 \end{vmatrix} =$$

Property IV. If we subtract a scalar multiple of any row from another row (or a scalar multiple of any column from another column), then the determinant of the new matrix is still $|A|$

Property V. If one row of A is a multiple of another row of A (or one column of A is a multiple of another column A), then $|A| = 0$

Example 7. What is $\begin{vmatrix} 3 & 1 & 0 \\ 6 & 2 & 0 \\ 1 & 0 & 3 \end{vmatrix}$?

- A square matrix A is **upper triangular** if $A_{ij} = 0$ for $i > j$:

- A square matrix A is **lower triangular** if $A_{ij} = 0$ for $i < j$:

Property VI. If A is an upper triangular or lower triangular matrix, then $|A|$ is the product of the diagonal entries: $|A| = A_{11}A_{22}\ldots A_{nn}$.

Example 8. Find $\begin{vmatrix} 2 & 7 & 0 & 1 \\ 0 & 6 & 4 & 8 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix}$.

6 Practice makes perfect

Example 9. Find the following determinants:

a. $\begin{vmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{vmatrix}$ b. $\begin{vmatrix} 8 & 3 & 0 \\ 3 & 4 & 5 \\ -1 & 0 & 7 \end{vmatrix}$ c. $\begin{vmatrix} 2 & 8 & 0 \\ 0 & 3 & 5 \\ 6 & -1 & 7 \end{vmatrix}$ d. $\begin{vmatrix} 2 & 3 & 8 \\ 0 & 4 & 3 \\ 6 & 0 & -1 \end{vmatrix}$

7 Cramer's rule

- Suppose we want to solve a system of equations $AX = B$ for X, where A is $n \times n$ and B is $n \times 1$
  - Quick check: X has dimension
- Let A_j be the matrix A, but with the jth column replaced by B
- **Cramer's rule:**

Example 10. Solve the following system of equations using Cramer's rule:

$$\begin{aligned} 2x_1 + 3x_2 \phantom{{}+5x_3} &= 8 \\ 4x_2 + 5x_3 &= 3 \\ 6x_1 \phantom{{}+4x_2} + 7x_3 &= -1 \end{aligned}$$